Appendix B

Formula Sheet

My formula sheets were pretty thorough, but perhaps they were missing something.
–Richard Rusczyk

B.1 Standard Formulas

Throughout this paper, $\triangle ABC$ is a triangle with vertices in counterclockwise order. The lengths will be abbreviated $a = BC$, $b = CA$, $c = AB$. These correspond with points in the vector plane $\vec{A}$, $\vec{B}$, $\vec{C}$.

For arbitrary points P, Q, R, $[PQR]$ will denote the signed area of $\triangle PQR$.[1]

Definition. Each point in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1$$

Theorem 1 (Line)**.** *The equation of a line is $ux + vy + wz = 0$ where u, v, w are reals. (These u, v and w are unique up to scaling.)*

Coordinates of Special Points

From this point on, the point $(kx : ky : kz)$ will refer to the point (x, y, z) for $k \neq 0$. In fact, the equations for the line and circle are still valid; hence, when one is simply intersecting lines and circles, it is permissible to use these un-homogenized forms in place of their normal forms. Again, the coordinates here are not homogenized!

Point	Coordinates	Sketch of Proof
Centroid	$G = (1 : 1 : 1)$	Trivial
Incenter	$I = (a : b : c)$	Angle bisector theorem
Symmedian point	$K = (a^2 : b^2 : c^2)$	Similar to above
Excenter	$I_a = (-a : b : c)$, etc.	Similar to above
Orthocenter	$H = (\tan A : \tan B : \tan C)$	Use area definition
Circumcenter	$O = (\sin 2A : \sin 2B : \sin 2C)$	Use area definition

If absolutely necessary, it is sometimes useful to convert the trigonometric forms of H and O into expressions entirely in terms of the side lengths (cf. [3, 5]) by

$$O = (a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$$

[1] For ABC counterclockwise, this is positive when P, Q and R are in counterclockwise order, and negative otherwise. When ABC is labeled clockwise the convention is reversed; that is, $[PQR]$ is positive if and only if it is oriented in the same way as ABC. In this article, ABC will always be labeled counterclockwise.

and

$$H = ((a^2+b^2-c^2)(c^2+a^2-b^2) : (b^2+c^2-a^2)(a^2+b^2-c^2) : (c^2+a^2-b^2)(b^2+c^2-a^2))$$

Definition. The *displacement vector* of two (normalized) points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is denoted by $\overrightarrow{PQ}$ and is equal to $(p_1 - q_1, p_2 - q_2, p_3 - q_3)$.

A Note on Scaling Displacement Vectors

In EFFT, one can write a displacement vector (x, y, z) as $(kx : ky : kz)$, and the theorem will still be true. This is also true for Strong EFFT, but NOT for the distance formula.

Theorem 4 (Evan's Favorite Forgotten Trick)**.** *Consider displacement vectors* $\overrightarrow{MN} = (x_1, y_1, z_1)$ *and* $\overrightarrow{PQ} = (x_2, y_2, z_2)$*. Then* $MN \perp PQ$ *if and only if*

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)$$

Corollary 5. *Consider a displacement vector* $\overrightarrow{PQ} = (x_1, y_1, z_1)$*. Then* $PQ \perp BC$ *if and only if*

$$0 = a^2(z_1 - y_1) + x_1(c^2 - b^2)$$

Corollary 6. *The perpendicular bisector of* BC *has equation*

$$0 = a^2(z - y) + x(c^2 - b^2)$$

Theorem 7 (Distance Formula)**.** *Consider a displacement vector* $\overrightarrow{PQ} = (x, y, z)$*. Then*

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy$$

Theorem 8. *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for reals u, v, w*.*

Corollary 9. *The circumcircle has equation*

$$a^2yz + b^2zx + c^2xy = 0$$

Theorem 10 (Area Formula)**.** *The area of a triangle with vertices* $P = (x_1, y_1, z_1)$*,* $Q = (x_2, y_2, z_2)$ *and* $R = (x_3, y_3, z_3)$ *is*

$$[PQR] = [ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Corollary 11 (First Collinearity Criteria)**.** *The points* $P = (x_1 : y_1 : z_1)$*,* $Q = (x_2 : y_2 : z_2)$ *and* $R = (x_3 : y_3 : z_3)$ *are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Corollary 12 (Line Through 2 Points)**.** *The equation of a line through the points $P = (x_1 : y_1 : z_1)$ and $Q = (x_2 : y_2 : z_2)$ is*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0$$

Corollary 13 (Second Collinearity Criteria)**.** *The points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$, are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Cyclic variations hold.

Theorem 14 (Strong EFFT)**.** *Suppose M, N, P and Q are points with*

$$\overrightarrow{MN} = x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO}$$
$$\overrightarrow{PQ} = x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}$$

If either $x_1 + y_1 + z_1 = 0$ or $x_2 + y_2 + z_2 = 0$, then $MN \perp PQ$ if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)$$

Corollary 15. *The equation for the tangent to the circumcircle at A is $b^2z + c^2y = 0$.*

Definition (Conway's Notation)**.** Let S be twice the area of the triangle. Define $S_\theta = S\cot\theta$, and define the shorthand notation $S_{\theta\phi} = S_\theta S_\phi$.

Fact. We have $S_A = \frac{-a^2+b^2+c^2}{2} = bc\cos A$ and its cyclic variations. (We also have $S_\omega = \frac{a^2+b^2+c^2}{2}$, where ω is the Brocard angle. This follows from $\cot\omega = \cot A + \cot B + \cot C$.)

Fact. We have the identities

$$S_B + S_C = a^2$$

and

$$S_{AB} + S_{BC} + S_{CA} = S^2$$

Fact. $O = (a^2S_A : b^2S_B : c^2S_C)$ and $H = (S_{BC} : S_{CA} : S_{AB}) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$.

Theorem 16 (Conway's Formula)**.** *Given a point P with counter-clockwise directed angles $\measuredangle PBC = \theta$ and $\measuredangle BCP = \phi$, we have $P = (-a^2 : S_C + S_\phi : S_B + S_\theta)$.*

Lemma 17 (Parallelogram Lemma)**.** *The points $ABCD$ form a parallelogram iff $A + C = B + D$ (here the points are normalized), where addition is done component-wise.*

Lemma 18 (Concurrence Lemma)**.** *The three lines $u_ix + v_iy + w_iz = 0$, for $i = 1, 2, 3$ are concurrent if and only if*

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$$

B.2 More Obscure Formulas

Here's some miscellaneous formulas and the like. These were not included in the main text.

From [7],

Theorem 19 (Leibniz Theorem)**.** *Let Q be a point with homogeneous barycentric coordinates $(u : v : w)$ with respect to $\triangle ABC$. For any point P on the plane ABC the following relation holds:*

$$uPA^2 + vPB^2 + wPC^2 = (u + v + w)PQ^2 + uQA^2 + vQB^2 + wQC^2$$

B.2.1 Other Special Points

Point	Coordinates
Gregonne Point [3]	$\text{Ge} = ((s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b))$
Nagel Point [3]	$\text{Na} = (s-a : s-b : s-c)$
Isogonal Conjugate [1]	$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$
Isotomic Conjugate [1]	$P^t = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$
Feuerbach Point [8]	$F = \left((b+c-a)(b-c)^2 : (c+a-b)(c-a)^2 : (a+b-c)(a-b)^2\right)$
Nine-point Center	$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B))$

B.2.2 Special Lines and Circles

Nine-point Circle	$-a^2yz - b^2xz - c^2xy + \frac{1}{2}(x+y+z)(S_Ax + S_By + S_Cz) = 0$
Incircle	$-a^2yz - b^2zx - c^2xy + (x+y+z)\left((s-a)^2x + (s-b)^2y + (s-c)^2z\right) = 0$
A-excircle [9]	$-a^2yz - b^2zx - c^2xy + (x+y+z)\left(s^2x + (s-c)^2y + (s-b)^2z\right) = 0$
Euler Line [9]	$S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z = 0$